ON A TYPE OF VARIATIONAL PROBLEMS

(OB ODNOM TIPE VARIATSIONNYKH ZADACH)

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In this work a class of problems of the calculus of variations is considered, and an application of the developed theory to multistage rockets is given.

In contrast to [1-3], the control functions $u_j(t)$ are here assumed to be known discontinuous functions of time.

A problem of the Bolza-Mayer type in the calculus of variations is posed and necessary conditions (conditions of stability) are derived for the determination of the points t_i of discontinuity of the control function $u_j(t)$ in order to obtain the extrema of certain functionals.

1. A process which takes place in a certain dynamical system is described by means of n ordinary first order differential equations

$$g_s = \dot{x}_s - f_s(x_1, \ldots, x_n, u_1, \ldots, u_m, t) = 0, \qquad (s = 1, \ldots, n) \qquad (1.1)$$

and by a finite set of relationships

$$\psi_k = \psi_k(u_1, \ldots, u_m, t) = 0 \qquad (k = 1, \ldots, r < m) \qquad (1.2)$$

In equations (1.1) and (1.2) the $x_s(t)$ are the coordinates which determine the position of the dynamical system, while the $u_j(t)$ are the discontinuous control functions; $m - r$ of them (say, $u_1, \ldots, u_{m-r}$) are given as explicit functions of time.

At the initial moment, $t = t_0$, the position of the system is determined by the values of the coordinates

$$x_s(t_0) = x_s^\circ \qquad (s = 1, \ldots, n) \qquad (1.3)$$

and by the values of the control functions

$$u_j(t_0) = u_j^\circ \qquad (j = 1, \ldots, m) \tag{1.4}$$

The coordinates at the time $t = T$ are related by the equations

$$\Phi_l = \Phi_l\,[x_s(T), T] = 0, \quad (l = 1, \ldots, p < n) \tag{1.5}$$

It is required to determine the moments of time t_i which determine an extremum to a certain functional

$$J = J\,[x_s(T), T] \tag{1.6}$$

under conditions (1.1) to (1.5).

2. We form, as is usually done [1,3], the auxiliary functional

$$I = J + \sum_{l=1}^{p} \rho_l \Phi_l + \int_{t_0}^{T} \left\{ \sum_{s=1}^{n} \lambda_s g_s + \sum_{k=1}^{r} \mu_k \psi_k \right\} dt \tag{2.1}$$

Here, $\lambda_s(t)$, $\mu_k(t)$ and ρ_l are the undetermined Lagrange multipliers. Since the right-hand side differs from J and only by terms which vanish at an extremum, the conditions for extrema of I and J coincide.

In evaluating the variation of the right-hand side of equation (2.1) we assume that in the time interval (t_0, T) considered, there exist two points t_1 and t_2 where the functions $u_j(t)$ are discontinuous

$$t_0 < t_1 < t_2 < T$$

The values of the above introduced functions we shall label with a superscript 1 in the interval (t_0, t_1) (for example, we write $x_s^{(1)}(t)$, $u_j^{(1)}(t)$; in the interval (t_1, t_2) we shall label them with the superscript 2 (for example, $x_s^{(2)}(t)$, $u_j^{(2)}(t)$), and in the interval (t_2, T) with the superscript 3.

Then the variation of expression (2.1) can be represented in the form

$$\Delta I = \Delta J + \Delta \sum_{l=1}^{p} \rho_l \Phi_l + \delta \int_{t_0}^{t_1} \left\{ \sum_{s=1}^{n} \lambda_s^{(1)} g_s^{(1)} + \sum_{n=1}^{r} \mu_k^{(1)} \psi_k^{(1)} \right\} dt +$$

$$+ \delta \int_{t_1}^{t_2} \left\{ \sum_{s=1}^{n} \lambda_s^{(2)} g_s^{(2)} + \sum_{k=1}^{r} \mu_k^{(2)} \psi_k^{(2)} \right\} dt + \delta \int_{t_2}^{T} \left\{ \sum_{s=1}^{n} \lambda_s^{(3)} g_s^{(3)} + \sum_{k=1}^{r} \mu_k^{(3)} \psi_k^{(3)} \right\} dt \tag{2.2}$$

The presence of the discontinuities in the control functions, forces us to consider the changes in the points, t_i $(i = 1, 2)$, of discontinuity when we compute the variation of the functional ΔI.

If the time is not fixed (T is free) then there exists the relation

$$\Delta x_s^{(3)}(T) = \delta x_s^{(3)}(T) + \dot{x}_s^{(3)}(T)\,\delta T \tag{2.3}$$

between the "variation of the end" $\Delta x_s^{(3)}(T)$ and the "variation at the end" $\delta x_s^{(3)}(T)$.

The variation ΔJ can be expressed in the form

$$\Delta J = \sum_{s=1}^{n} \frac{\partial J}{\partial x_s^{(3)}(T)} \delta x_s^{(3)}(T) + \left[\frac{\partial J}{\partial T} + \sum_{s=1}^{n} \frac{\partial J}{\partial x_s^{(3)}(T)} \dot{x}_s^{(3)}(T) \right] \delta T \tag{2.4}$$

For the variation of $\rho_1 \Phi_1 + \ldots + \rho_p \Phi_p$ one obtains an analogous equation.

Since the initial data, which determine the position of the dynamical system, are known ((1.3) to (1.4)) and the control functions are also determined on the interval (t_0, t_i), then, in accordance with the formulation of the problem, the function $x_s^{(1)}(t)$ of time can be determined from the corresponding system of equations, and, hence, the variation of the first integral in expression (2.2) must be zero. For the same reason

$$\delta x_s^{(1)}(t_1) = 0 \tag{2.5}$$

namely, the "variation of the end" of the left trajectory may have an increment along this trajectory, and hence the equation, similar to equation (2.3), will have the form

$$\Delta x_s^{(1)}(t_1) = \dot{x}_s^{(1)}(t_1)\, \delta t_1 \tag{2.6}$$

The values of the "variation at the end" $\delta x_s^{(2)}(t_1)$ and of the "variation of the end" $\Delta x_s^{(2)}(t_1)$ of the intermediate trajectory are related by the equation

$$\Delta x_s^{(2)}(t_1) = \delta x_s^{(2)}(t_1) + \dot{x}_s^{(2)}(t_1)\, \delta t_1 \tag{2.7}$$

Because of the discontinuity of the trajectory $x_s(t)$ at the points of the discontinuities of the control functions $u_j(t)$, we have

$$\Delta x_s^{(1)}(t_1) = \Delta x_s^{(2)}(t_1) = \Delta x_s(t_1) \tag{2.8}$$

From equations (2.6) to (2.8) it follows that

$$\delta x_s^{(2)}(t_1) = [\dot{x}_s^{(1)}(t_1) - \dot{x}_s^{(2)}(t_1)]\, \delta t_1 \tag{2.9}$$

For the variation of the right trajectory $x_s^{(3)}(t)$ at the point t_2 we obtain

$$\delta x_s^{(3)}(t_2) = \delta x_s^{(2)}(t_2) + [\dot{x}_s^{(2)}(t_2) - \dot{x}_s^{(3)}(t_2)]\, \delta t_2 \tag{2.10}$$

Taking into account equations (2.3) to (2.10), we can transform the variation of the functional ΔI into the following form

$$\Delta I = \sum_{s=1}^{n} \frac{\partial J}{\partial x_s^{(3)}(T)} \delta x_s^{(3)}(T) + \left[\frac{\partial J}{\partial T} + \sum \frac{\partial J}{\partial x_s^{(3)}(T)} \dot{x}_s^{(3)}(T)\right] \delta T +$$
$$+ \Delta \sum_{l=1}^{p} \rho_l \Phi_l - \left[\sum_{s=1}^{n} \lambda_s^{(2)}(t_1)\left(\dot{x}_s^{(1)}(t_1) - \dot{x}_s^{(2)}(t_1)\right)\right] \delta t_1 + \Sigma \lambda_s^{(3)}(T)\, \delta x_s^{(3)}(T) +$$
$$+ \sum_{s=1}^{n} [\lambda_s^{(2)}(t_2) - \lambda_s^{(3)}(t_2)]\, \delta x_s^{(2)}(t_2) - \left[\sum_{s=1}^{n} \lambda_s^{(3)}(t_2)\, (\dot{x}_s^{(2)}(t_2) - \dot{x}_s^{(3)}(t_2))\right] \delta t_2 +$$
$$+ \int_{t_1}^{t_2} \Big\{ \sum_{s=1}^{n} \delta\lambda_s^{(2)} [\dot{x}_s^{(2)} - f_s(x_1^{(2)}, \ldots, x_n^{(2)}, u_1^{(2)}, \ldots, u_m^{(2)}, t)] +$$
$$+ \sum_{k=1}^{r} \delta\mu_k^{(2)} \psi_k(u_1^{(2)}, \ldots, u_m^{(2)}, t) \Big\} dt - \int_{t_1}^{t_2} \Big\{ \sum_{s=1}^{n} \delta x_s^{(2)} \left[\dot{\lambda}_s^{(2)} + \sum_{\alpha=1}^{n} \frac{\partial f_\alpha}{\partial x_s^{(2)}} \lambda_\alpha^{(2)}\right] +$$
$$+ \sum_{k=1}^{m} \delta u_k^{(2)} \left[\sum_{s=1}^{n} \lambda_s^{(2)} \frac{\partial f_s}{\partial u_k^{(2)}} - \sum_{\beta=1}^{r} \mu_\beta^{(2)} \frac{\partial \psi_\beta}{\partial u_k^{(2)}}\right]\Big\} dt +$$
$$+ \int_{t_2}^{T} \Big\{ \sum_{s=1}^{n} \delta\lambda_s^{(3)} [\dot{x}_s^{(3)} - f_s(x_1^{(3)}, \ldots, x_n^{(3)}, u_1^{(3)}, \ldots, u_m^{(3)}, t)] +$$
$$+ \sum_{k=1}^{r} \delta\mu_k^{(3)} \psi_k(u_1^{(3)}, \ldots, u_m^{(3)}, t) \Big\} dt - \int_{t_2}^{T} \Big\{ \sum_{s=1}^{n} \delta x_s^{(3)} \left[\dot{\lambda}_s^{(3)} + \sum_{\alpha=1}^{n} \frac{\partial f_\alpha}{\partial x_s^{(3)}} \lambda_\alpha^{(3)}\right] +$$
$$+ \sum_{k=1}^{m} \delta u_k^{(3)} \left[\sum_{s=1}^{n} \lambda_s^{(3)} \frac{\partial f_s}{\partial u_k^{(3)}} - \sum_{\beta=1}^{r} \mu_\beta^{(3)} \frac{\partial \psi_\beta}{\partial u_k^{(3)}}\right]\Big\} dt \qquad (2.11)$$

In the derivation of formula (2.11) we have made use of the equations

$$\int_{t_1}^{t_2} \sum_{s=1}^{n} \lambda_s^{(2)} \delta\dot{x}_s^{(2)}\, dt = \sum_{s=1}^{n} \lambda_s^{(2)}(t_2)\, \delta x_s^{(2)}(t_2) - \sum_{s=1}^{n} \lambda_s^{(2)}(t_1)\, \delta x_s^{(2)}(t_1) -$$
$$- \int_{t_1}^{t_2} \sum_{s=1}^{n} \dot{\lambda}_s^{(2)} \delta x_s^{(2)}\, dt \qquad (2.12)$$
$$\int_{t_2}^{T} \sum_{s=1}^{n} \lambda_s^{(3)} \delta\dot{x}_s^{(3)}\, dt = \sum_{s=1}^{n} \lambda_s^{(3)}(T)\, \delta x_s^{(3)}(T) - \sum_{s=1}^{n} \lambda_s^{(3)}(t_2)\, \delta x_s(t_2) - \int_{t_2}^{T} \sum_{s=1}^{n} \dot{\lambda}_s^{(3)} \delta x_s^{(3)}\, dt$$

The type of the function $u_j(t_i, T, t)$, which is determined in any concrete problem by the conditions of that problem, does not affect the procedure of the proof. Therefore, for the sake of definiteness, we shall use $u_j^{(2)}(t, t_1)$ and $u_j^{(3)}(t, t_1, t_2)$. Then the variations of the control function $u_j(t)$ will have the form

$$\delta u_j^{(2)} = \frac{\partial u_j^{(2)}}{\partial t_1}\delta t_1, \qquad \delta u_j^{(3)} = \frac{\partial u_j^{(3)}}{\partial t_1}\delta t_1 + \frac{\partial u_j^{(3)}}{\partial t_2}\delta t_2 \tag{2.13}$$

The variations

$$\delta x_s^{(2)}(t),\ \delta x_s^{(3)}(t),\ \delta x_s^{(2)}(t_2),\ \delta\lambda_s^{(2)}(t),\ \delta\lambda_s^{(3)}(t),\ \delta\mu_k^{(2)}(t),\ \delta\mu_k^{(3)}(t)$$
$$(k = 1, \ldots, r),\ \ \delta t_1,\ \delta t_2,\ \delta T,\ n - p$$

of the variations $\delta_s(T)$ are independent.

By determining the $2r$ multipliers $\mu_k^{(2)}(t)$, $\mu_k^{(3)}(t)$ so that the coefficients of the variations $\delta_\nu^{(2)}$ and $\delta_\nu^{(3)}$ ($\nu = m - r + 1, \ldots, m$) are zero, and determining the p multipliers ρ_l so that the coefficients of the dependent variations $\delta_s^{(3)}(T)$ are zero, we make sure that the coefficients of the remaining independent variations are also zero. As a result we obtain the equations which must be satisfied by the coordinates of the system and by the control functions

$$\dot{x}_s^i = f_s^i(x^i, u^i, t) \qquad \psi_k^i(u^i, t) = 0, \qquad (s = 1, \ldots, n;\ k = 1, \ldots, r;\ i = 2, 3) \tag{2.14}$$

The differential equations which are satisfied by the functions $\lambda_s^i(t)$ are the following:

$$\dot{\lambda}_s^i + \sum_{\alpha=1}^{n} \lambda_\alpha^i \frac{\partial f_\alpha}{\partial x_s^i} = 0 \qquad (s = 1, \ldots, n;\ i = 2,3) \tag{2.15}$$

The boundary conditions for the functions $\lambda_s^{(3)}(T)$ are

$$\lambda_s^{(3)}(T) + \frac{\partial}{\partial x_s^{(3)}(T)}\left[J + \sum_{l=1}^{p} \rho_l \Phi_l\right] = 0 \tag{2.16}$$

The boundary condition is of the form

$$\frac{d}{dT}\left[J + \sum_{l=1}^{p} \rho_l \Phi_l\right] = 0 \tag{2.17}$$

The conditions of continuity for the functions $\lambda_s(t)$ are

$$\lambda_s^{(2)}(t_2) = \lambda_s^{(3)}(t_2) \qquad (s = 1, \ldots, n) \tag{2.18}$$

The equations for the determination of the multipliers $\mu_k^{(2)}(t)$ and $\mu_k^{(3)}(t)$ are

$$\sum_{\alpha=1}^{n} \lambda_\alpha^i \frac{\partial f_\alpha}{\partial u_k^i} - \sum_{k=1}^{r} \mu_k^i \frac{\partial \psi_k}{\partial u_k^i} = 0 \qquad (k = 1, \ldots, r;\ \ i = 2, 3) \tag{2.19}$$

For the points of discontinuity t_1 and t_2 of the control function

$u_j(t)$, we have the following conditions:

$$\sum_{s=1}^{n} \lambda_s^{i+1}(t_i)\,[\dot{x}_s^{\,i}(t_i) - \dot{x}_s^{\,i1}\ \ (t_i)] + \int_{t_i}^{T} \left\{ \sum_{k=1}^{m-r} \left[\sum_{s=1}^{n} \lambda_s \frac{\partial f_s}{\partial u_k} - \sum_{\beta=1}^{r} \mu_\beta \frac{\partial \psi_\beta}{\partial u_k} \right] \frac{\partial u_k}{\partial t_i} \right\} \partial t = 0 \quad (2.20)$$

Thus we have obtained the following:

$2n$ first order differential equations for the determination of the functions

$$x_s^{(2)}(t),\ x_s^{(3)}(t)\ \ (s = 1, \ldots, n)$$

$2n$ first order differential equations for the determination of the multipliers

$$\lambda_s^{(2)}(t),\ \lambda_s^{(3)}(t)\ \ (s = 1, \ldots, n)$$

$2r$ relations for determining the functions

$$\mu_k^{(2)}(t), \mu_k^{(3)}(t)\ (k = 1, \ldots, r)$$

The quantities which are still unknown are $4n$ arbitrary constants, obtained in solving the corresponding first order differential equations, the quantities t_1, t_2 and T, and also the p multipliers ρ_l ($l = 1, \ldots, p$); altogether $4n + p + 3$ quantities.

For the determination of these unknown quantities we have the n boundary conditions (2.16), n conditions (2.18) of the continuity of the multipliers $\lambda_s(t)$, two conditions (2.20) at the point t_2 of discontinuity of the control, n conditions of continuity of the coordinates at the point t_1, $x_s^{(1)}(t_1) = x_s^{(2)}(t_1)$, and n conditions of the continuity of the coordinates at the point t_2, $x_s^{(2)}(t_2) = x_s^{(3)}(t_2)$, and the p relations (1.5); altogether $4n + p + 3$ conditions.

Therefore, the problem of determining the extremum of the functional J can be solved.

We note, that following [1,3], one can introduce a Lagrange function L given by the equation

$$L = \sum_{s=1}^{n} \lambda_s g_s + \sum_{k=1}^{r} \mu_k \psi_k$$

The system of differential equations (2.15), which determines the multipliers $\lambda_s^{\,i}(t)$ ($s = 1, \ldots, n$; $i = 2, 3$) can be expressed in the form

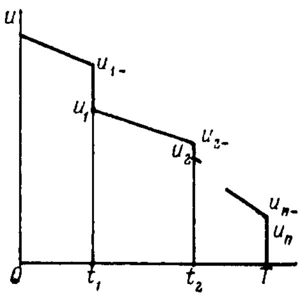
$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_s^{\,i}}\right] - \frac{\partial L}{\partial x_s^{\,i}} = 0 \qquad (s = 1, \ldots, n)$$

while conditions (2.18) of the continuity of the functions $\lambda_s(t)$ can be written as

$$\left[\frac{\partial L}{\partial \dot{x}_s^{(2)}}\right]_{t_1-0} = \left[\frac{\partial L}{\partial \dot{x}_s^{(3)}}\right]_{t_2+0}$$

Let us introduce the function H given by the equation

$$H = \sum_{s=1}^{n} \lambda_s f_s - \sum_{k=1}^{r} \mu_k \psi_k = H_\lambda + H_\mu$$



where

$$H=\lambda_\lambda \sum_{s=1}^{n} \lambda\ f_s, \qquad H_\mu = -\sum_{k=1}^{r} \mu_k \psi_k$$

Taking into account the fact that

$$\frac{\partial L}{\partial u_k} = \frac{\partial H}{\partial u_k}$$

we may write conditions (2.20) in the form

$$[H_\lambda]_{t_i-0} - [H_\lambda]_{t_i+0} + \int_{t_i}^{T} \left[\sum_{k=1}^{m-r} \frac{\partial H}{\partial u_k}\frac{\partial u_k}{\partial t_i}\right] dt = 0 \, (i = 1, 2)$$

Next, using the equation

$$\sum_{k=1}^{m-r} \frac{\partial H}{\partial u_k}\frac{\partial u_k}{\partial t_i} = \frac{\partial H}{\partial t_i}$$

we express (2.2) finally in the form

$$[H_\lambda]_{t_i-0} - [H_\lambda]_{t_i+0} + \int_{t_i}^{T} \frac{\partial H}{\partial t_i} dt = 0 \qquad (i = 1, 2)$$

3. As an example let us consider the application of the theory presented above to the computation of a two-stage rocket which is moving vertically in a nonhomogeneous gravitational field in a space where aerodynamic forces can be neglected.

The equations of motion of the center of mass of the composite rocket have the form

$$\dot{v} = -g - V^r \frac{\dot{m}}{m}, \qquad \dot{h} = v, \qquad g = g_0\left(1 - \frac{2h}{R}\right) \tag{3.1}$$

where m is the mass of the composite rocket which changes according to a linear law, v is the velocity of the center of mass of the composite rocket, h is the altitude above the earth's surface, V^r is the relative velocity of the expelled particles, g is the acceleration of gravity, R is the earth's radius and g_0 is the acceleration of gravity at the earth's surface, where $h = 0$.

Let us introduce a dimensionless mass for the multistage rocket

$$u = \frac{m}{m_0} \tag{3.2}$$

where m_0 is the starting mass of the rocket.

If one considers an auxiliary plane $\{u, t\}$, then the parts of the curve $u = u(t)$ which correspond to the operating regime of the engines will be represented by inclined line segments while the parts of the separation of the stages will be vertical segments (see Figure).

The engines of the successive subrockets work without intermissions.

Let us denote the ratio of the "mass of the dry weight" of the ith ($i = 1, 2$) stage to the mass of its fuel by k_i. Furthermore, let us use the notation

$$u_{i-} = u(t_i - 0), \qquad u_i = u(t_i + 0) \tag{3.3}$$

The function $u(t)$ is determined analytically by $2n$ equations ($n = 2$)

$$u_{i-} = u_{i-1} - \beta_i(t_i - t_{i-1}), \qquad u_i = u_{i-}(1 + k_i) - k_i u_{i-1} \tag{3.4}$$

where β_i is the fuel expenditure per second of the engine of the ith stage.

We note that $t_0 = 0$, $t_2 = T$ and $u_2 = m_p/m_0$ (m_p is the mass of the useful load).

For what follows it is necessary to evaluate the partial derivatives

$$\frac{\partial u_j}{\partial t_i}, \qquad \frac{\partial u_{j-}}{\partial t_i} \qquad (j = 1, 2;\ i = 1)$$

For a two-stage rocket the evaluations yield

$$\frac{\partial u_{1-}}{\partial t_1} = -\beta_1, \qquad \frac{\partial u_{2-}}{\partial t_1} = -\beta_2 k_2, \qquad \frac{\partial u_1}{\partial t_1} = -\beta_1(1 + k_1) \tag{3.5}$$

$$\frac{\partial u_2}{\partial t_1} = \beta_2(1 + k_2) - \beta_1(1 + k_1) = 0 \tag{3.6}$$

It is required to find the time moment t_1 of the transition from the first stage of the composite rocket when the velocity $v(T)$ at the end of the active part has reached its maximum, if the active flight time of the rocket is fixed (specified).

The differential equations (2.15) for the determination of the undetermined multipliers $\lambda_1(t)$ and $\lambda_2(t)$ take on the form

$$\dot{\lambda}_1 = -\lambda_2, \qquad \dot{\lambda}_2 = -\nu^2\lambda_1 \tag{3.7}$$

where

$$\nu^2 = 2g_0/R \tag{3.8}$$

Boundary conditions (2.16) are (h is free)

$$\lambda_1(T) = -1, \qquad \lambda_2(T) = 0 \tag{3.9}$$

The values $\lambda_1(t)$ and $\lambda_2(t)$, found by means of equations (3.7) and (3.9), have the form

$$\lambda_1 = \cosh\nu\,(T-t), \qquad \lambda_2 = \nu\sinh\nu\,(T-t) \tag{3.10}$$

Condition (2.20) can be expressed in the form

$$\lambda_1(t_1)\left[\frac{\beta_1 V_1{}^r}{u_{1-}} - \frac{\beta_2 V_2{}^r}{u_1}\right] + \beta_2 k_2 V_2{}^r \int_{t_1}^{T} \frac{\lambda_1(t)}{u^2}\,dt = 0 \tag{3.11}$$

Integrating the last integral by parts, we can transform (3.11) to the form

$$\lambda_1(t_1)\left[\frac{\beta_1 V_1{}^r}{u_{1-}} - \frac{\beta_2 V_2{}^r(1+k_2)}{u_1}\right] + \frac{\beta_2^2 k_2 V_2{}^r}{u_{2-}} + \nu\beta_2{}^2 k_2 V_2{}^r \int_{t_1}^{T} \frac{\lambda_2(t)}{u}\,dt = 0 \tag{3.12}$$

For given values of β_i and $V_i{}^r$, equation (3.12) determines the required moment of time t_1.

If the rocket is uniform $\beta_1 = \beta_2 = \beta$, $V_1{}^r = V_2{}^r = V^r$ and $k_1 = k_2 = k$, then equation (3.12) takes on the form

$$\lambda_1(t_1)\left[\frac{1}{u_{1-}} - \frac{1+k}{u_1}\right] + \frac{k}{u_{2-}} + k\nu \int_{t_1}^{T} \lambda_2(t)\,\frac{dt}{u^2} = 0 \tag{3.13}$$

It should be mentioned that the computation of the optimal moment t_1 can be performed even if the law of motion of the center of mass of the rocket is not known.

As a numerical example, let us consider the determination of the time moment t_1 for a two-stage uniform rocket with the following characteristics:

$$m_0 = 1000 \text{ kg sec}^2\text{/meter}, \quad m_p = 50 \text{ kg sec}^2\text{/meter}, \quad k = 0.1,$$

$$g_0 = 10 \text{ meter/sec}, \quad V^r = 3000 \text{ meter/sec}; \quad \beta = 1/200 \text{ sec}^{-1}, \quad T = 190 \text{ sec}.$$

The optimal moment of time t_1 for the separation of the first stage, under the above hypothesis that the rocket is moving in a homogeneous gravitational field, in 156.2 seconds.

If the rocket is moving in a nonhomogeneous gravitational field, the quantity t_1 found by means of the equation (3.13) is 154 seconds.

The maximum velocity of the composite rocket at the end of the active regime in a homogeneous gravitational field is 5528 meter/sec, and in a nonhomogeneous field it is 5623 meter/sec.

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